Technical Advance: The Geometric-Series Solution (GSS) to spin up soil organic matter (SOM) pools

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Introduction

- Major application of SOM models:
  - Initialize SOM pools using a set of driving data
  - Apply another set of driving data to study the impact on SOM pools
- Pools of first-order based SOM models have equilibrium
  - Difficult to interpret the model results when SOM pools are not in equilibrium
  - The most intrinsic way to solve the equilibrum problem is to reiterate SOM models for numerous times to get the equilibrium values

How to spin up SOM pools?

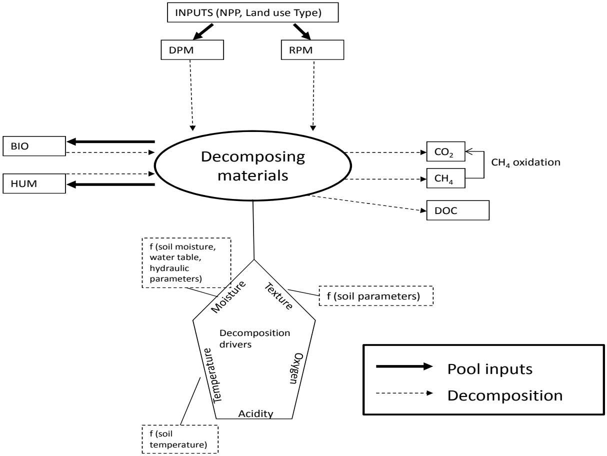
- Iterate the model
  - The computer is going to sweat!
  - Possible to use a longer time step (say, a year) to reduce the overhead, but it smooths the diurnal / seasonal fluctuation.
- Recent publications (Lardy et al., 2011 and Xia et al., 2012) suggested matrix methods to approximate the initialized SOM values.
  - The methods significantly reduce the number of iterations (still need some)
  - The methods are approximate methods, error analysis may be necessary

- Here we present an analytical method – the GSS method – which solves the spin-up problem
  - The solution is exact, no need for error analysis
  - Only one loop is required
  - Only implement one equation
- Major assumptions (limitations):
  - Users spin up SOM pools by repeating a limited set of driving data (e.g. you repeat a 30-year long-term average driving data for 100 cycles to simulate a 3000-year model run)
  - Modelled plant input and soil climate data do not change from one cycle to another cycle
  - Inputs to each SOM pool can be derived from plant inputs analytically. No gurantee that it is applicable to all SOM models, but yes for the JULES-RothC / JULES-ECOSSE model.
  - The SOM models are based on first-order-difference equations

$$\frac{\delta SOM_{pt}}{\delta t} = (Pool\_input_{pt} + SOM_{p(t-1)})(1 - e^{-k_{pt} l}) \tag{1}$$

$$k_{pt} = c_p \; x1_t \; x2_t \; \dots \; xb_t \tag{2}$$

Appendix: ECOSSE C components



Toolkits

Toolkit 1 – Sum of geometric series:

$$\sum_{n=1}^{N-1} e^{nx} = S - e^{0} = \frac{(1 - e^{nx})}{(1 - e^{x})} - 1 \tag{3}$$

Toolkit 2 – e^{-k_a}:

$$e^{-k_a} = e^{-k_1} e^{-k_2} \dots e^{-k_n} \tag{4}$$

Array of non-decomposed pool inputs during iteration

$$
\begin{array}{cccccc}
 & & & \text{j} & & \\
 & & 1 & 2 & \dots & \text{n} \\
 & 1 & \begin{matrix} I_{11}e^{-k_{11}}e^{-k_{12}}\dots e^{-k_{1n}} \\ e^{-k_{21}}e^{-k_{22}}\dots e^{-k_{2n}} \\ \vdots \\ e^{-k_{m1}}e^{-k_{m2}}\dots e^{-k_{mn}} \end{matrix} & \begin{matrix} I_{12}e^{-k_{12}}\dots e^{-k_{1n}} \\ e^{-k_{21}}e^{-k_{22}}\dots e^{-k_{2n}} \\ \vdots \\ e^{-k_{m1}}e^{-k_{m2}}\dots e^{-k_{mn}} \end{matrix} & \dots & \begin{matrix} I_{1n}e^{-k_{1n}} \\ e^{-k_{21}}e^{-k_{22}}\dots e^{-k_{2n}} \\ \vdots \\ e^{-k_{m1}}e^{-k_{m2}}\dots e^{-k_{mn}} \end{matrix} \\
\text{i} & & & & & \\
 & 2 & \begin{matrix} I_{21}e^{-k_{21}}e^{-k_{22}}\dots e^{-k_{2n}} \\ \vdots \\ e^{-k_{m1}}e^{-k_{m2}}\dots e^{-k_{mn}} \end{matrix} & \begin{matrix} I_{22}e^{-k_{22}}\dots e^{-k_{2n}} \\ \vdots \\ e^{-k_{m1}}e^{-k_{m2}}\dots e^{-k_{mn}} \end{matrix} & \dots & \begin{matrix} I_{2n}e^{-k_{2n}} \\ \vdots \\ e^{-k_{m1}}e^{-k_{m2}}\dots e^{-k_{mn}} \end{matrix} \\
 & \vdots & \vdots & \vdots & & \vdots \\
 & m & I_{m1}e^{-k_{m1}}e^{-k_{m2}}\dots e^{-k_{mn}} & I_{m2}e^{-k_{m2}}\dots e^{-k_{mn}} & \dots & I_{mn}e^{-k_{mn}}
\end{array}
$$

Derivation of column sum I

$$
\begin{aligned}
colSums(n) =& I_{mn}(e^{-k_{mn}}) \\
& + \dots \\
& + I_{2n}(e^{-k_{2n}})(e^{-k_{m1}} e^{-k_{m2}} \dots e^{-k_{mn}}) \\
& + I_{1n}(e^{-k_{1n}})(e^{-k_{21}} e^{-k_{22}} \dots e^{-k_{2n}}) \dots (e^{-k_{m1}} e^{-k_{m2}} \dots e^{-k_{mn}})
\end{aligned} \tag{5}
$$

As plant inputs and soil climate data are the same across spin-up cycles, drop the first subscript:

$$
\begin{aligned}
colSums(n) =& I_n(e^{-k_n}) \\
& + \dots \\
& + I_n(e^{-k_n})(e^{-k_1} e^{-k_2} \dots e^{-k_n}) \\
& + I_n(e^{-k_n})(e^{-k_1} e^{-k_2} \dots e^{-k_n}) \dots (e^{-k_1} e^{-k_2} \dots e^{-k_n})
\end{aligned} \tag{6}
$$

Factor $I_n(e^{-k_n})$ out and use Toolkit 2:

$$
colSums(n) = I_n(e^{-k_n})(1 + e^{-k_a} + e^{-2k_a} + \dots + e^{-mk_a}) \tag{7}
$$

Derivation of column sum II

Use Toolkit 1:

$$colSums(n) = I_n(e^{-k_n})((1 - e^{-(m+1)k_a})/(1 - e^{-k_a}) - 1) \quad (8)$$

Sum across columns:

$$SOM_n = \sum_{n=1}^{n} colSums(n) \quad (9)$$

Illustration with WFDEI data

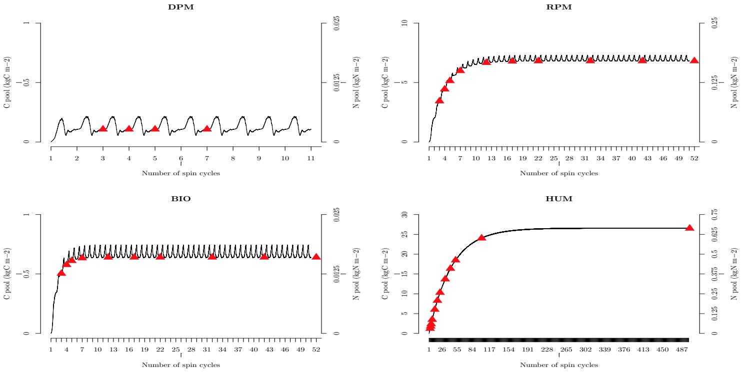


Figure: Iterative runs of SOM pools and the results calculated by the GSS method

Conclusion

- Based on mild assumptions, the GSS method was derived to initialize SOM pools
- Application:
  - Spin up the JULES model and generate modelled data of plant inputs and soil climate
  - Analytically find out the relationship between plant inputs and input of SOM pools
  - Apply the formulae to calculate column sum of a timestep across spin-up cycles, then sum across the timesteps to get the SOM pool value
  - $$colSums(j) = I_j(e^{-k_j}e^{-k_{(j+1)}}\dots e^{-k_n})(1-e^{-(m+1)k_a})/(1-e^{-k_a})-1 \quad (10)$$