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Analytical determination of the carbon cycle equilibrium in coupled climate and carbon cycle models

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Outline

- Motivation
- Some maths background
- Application
- Results
- Summary
- Outlook

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Motivation

- Carbon cycle models: predict future carbon (C) fluxes
- Simulation started with some initial state of the C pools (C in live biomass, litter, soil)
- Simulation results should reflect the response of the C system to the environmental variability during the simulation period
- Selection of initial state of C pools is crucial to avoid spurious C fluxes that result from adjustment of the system to “inappropriate” initial state
- Initial state often chosen as steady state of system (equilibrium)

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Motivation

- **Equilibrium**
- **Difficult** to determine for a complex system
- Run model with the same boundary conditions and forcing for a long time and wait for convergence to steady state (100s to 1000s of years): **Expensive**
- Not a good initial value for simulations from now into the future, rather for, say 19th century: simulation into the future needs to capture current disequilibrium of the C system from the past 150-200 years
- Need another 150-200 years simulation years - expensive

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Motivation

- **Equilibrium**
- Expensive
- Limits exploration of carbon cycle system
- Data assimilation (data assimilation of net C flux needs well established C state of the system)

- Direct determination from system dynamics?
- Needs to be computationally efficient

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Maths background

- Carbon cycle systems are being represented by

$$\frac{dx}{dt}(t) = A(t,x) + g(t), x(t_0) = x_0, t \geq t_0$$

- Frequently linear:

$$\frac{dx}{dt}(t) = A(t)x(t) + g(t), x(t_0) = x_0, t \geq t_0$$

- x are the carbon pools, A environmental effects, g input, e.g. photosynthesis
- Soil respiration in JULES

$$d(\mathrm{DPM})/dt = \alpha_{dr}\Lambda_c - R_{\mathrm{DPM}}$$
$$d(\mathrm{RPM})/dt = (1-\alpha_{dr})\Lambda_c - R_{\mathrm{RPM}}$$
$$d(\mathrm{BIO})/dt = 0.46\cdot\beta R_s - R_{\mathrm{BIO}}$$
$$d(\mathrm{HUM})/dt = 0.54\cdot\beta R_s - R_{\mathrm{HUM}}$$

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Maths background

$$\frac{dx}{dt}(t) = A(t)x(t) + g(t), x(t_0) = x_0, t \geq t_0$$

- Solution:

$$x(t) = \Psi(t, t_0)x_0 + \int_{t_0}^{t} \Psi(t, s)\, g(s)ds$$

- $\Psi(t, t_0)$ is the state transition matrix

- For constant A:

$$\Psi(t, t_0) = exp(A(t - t_0)) = e^{A(t-t_0)}$$

- State transfer between time points

$$\Psi(t_2, t_0) = \Psi(t_2, t_1)\Psi(t_1, t_0)$$

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Maths background

- Equilibrium:
- Steady state such that after one year the C pools are the same as at the beginning of the year:

$$x_0 = x(t_0 + \omega) = \Psi(t_0 + \omega, t_0)x_0 + \int_{t_0}^{t_0+\omega} \Psi(t, s)\, g(s) ds$$

ω period of one year

- So, for a steady state:

$$x_0 = (Id - \Psi(t_0 + \omega, t_0))^{-1} \int_{t_0}^{t_0+\omega} \Psi(t_0 + \omega, s)\, g(s) ds$$

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Application

- Carbon cycle model is run time stepped with time step Δt
- A and g constant during time step: A_i and g_i

$$x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-s)}\, g(s)ds$$

- From one time step to the next:

$$x_{i+1} = e^{A_i \Delta t}x_i + g_i \int_{t_i}^{t_{i+1}} e^{A_i(t-s)}\, ds$$

- **Recursion**

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Application

- Resolve recursion:

$$\Psi(t_0 + \omega, t_0) = \prod e^{A_i \Delta t}$$

$$\int_{t_0}^{t_0+\omega} \Psi(t_0 + \omega, s)\, g(s) ds = \sum g_i \int_{t_i}^{t_{i+1}} e^{A_i (t-s)}\, ds$$

- Computationally very efficient:
  - For n C pools add only one *n x n* matrix and one *n* vector to model code for each grid cell and PFT.
  - No need to output additional data or save any additional data in file for later calculations.

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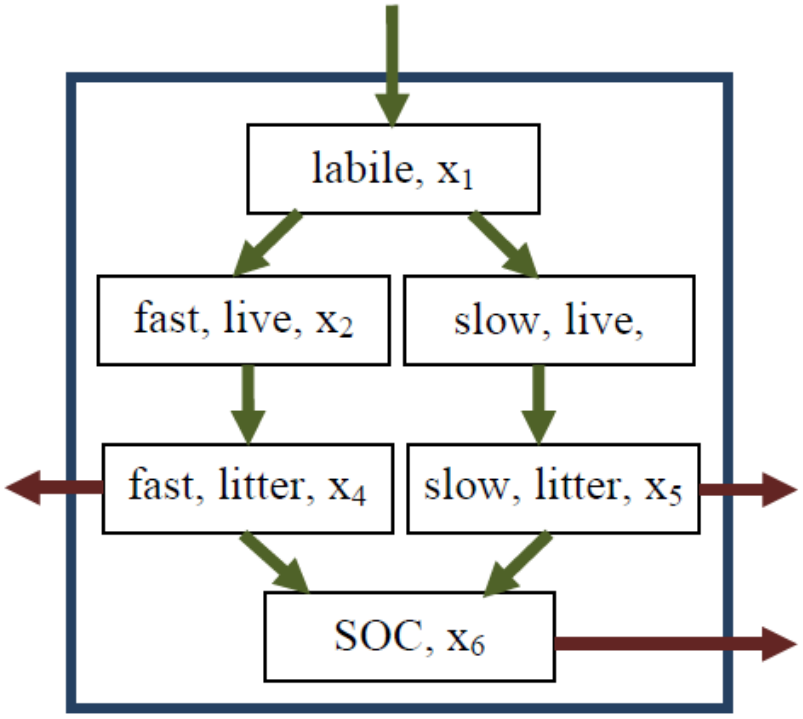
Application

Example

- Synthetic system

$$A = \begin{pmatrix} a_{11} & 0 & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 & 0 \\ a_{31} & 0 & a_{33} & 0 & 0 & 0 \\ 0 & a_{42} & 0 & a_{44} & 0 & 0 \\ 0 & 0 & a_{53} & 0 & a_{55} & 0 \\ 0 & 0 & 0 & a_{64} & a_{65} & a_{66} \end{pmatrix}$$



a_{11}=-abs(max(10^{-5},sin(t*pi/nt)/nt))
a_{22}=0.25*a_{11}; a_{21}=-0.5*a_{11}; a_{33}=1/16*a_{11}; a_{31}=-0.5*a_{11};

g(t)=5000*(max(0,sin(t*pi/nt)*sin(t*2*pi/(24*60)-pi/2))/nt 0 0 0 0 0)^T

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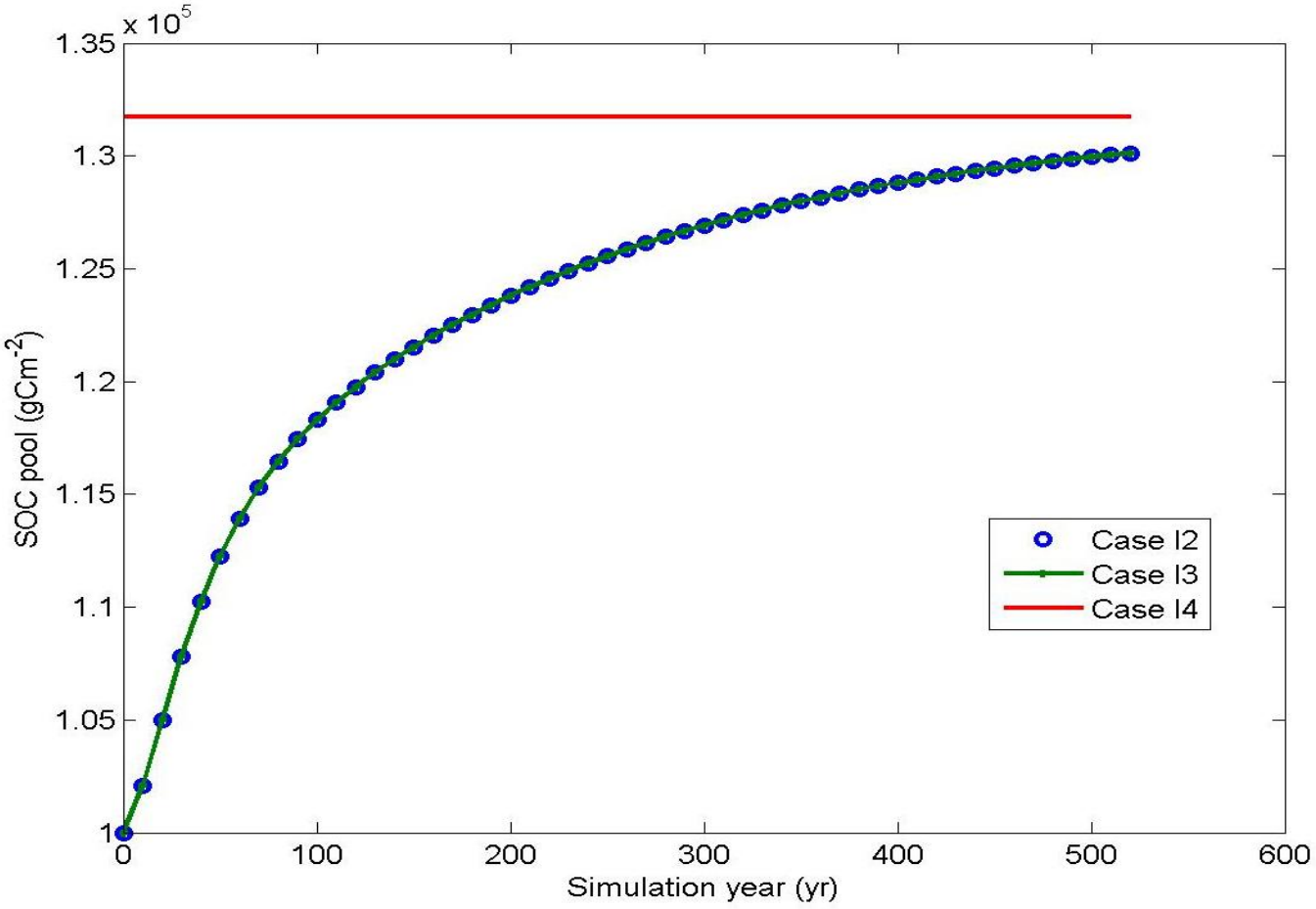
Results

Case	Short description	Simulation years $\Delta x_T<0.01$	cputime in matlab (s)	C_{Tot} von qss	C_{Tot} when $\Delta x_T<0.01$
I1	Numerical higher order solver, ode45	528	809	152883.70	151333.09
I2	Explicit forward iteration with A and g	528	336	152814.02	151269.73
I3	$\Psi(T,0)$ and $\tilde{x}$ determined using similarity transform, eig(T) and inv(T), iterated for all simulation years	528	2.4	152814.02	151269.72
I4	$\Psi(T,0)$ and $\tilde{x}$ determined using similarity transform, eig(T) and inv(T), solved for qss	1	2.4	152814.02	

Results

- "SOC" pool x_6



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Results

- Different methods deliver the same steady state
- New approach needs less than 1% of the time of other methods for steady state
- With a more computationally complex A and g (e.g. JULES) absolute time savings would be very significant

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Results

Non linear example

- Use linearization: Taylor-expansion

- Simple canopy integration:

$$p_0*(1-e^{-kLAI})/k$$

- p_0 top leaf photosynthesis
- Specific leaf area links to fast turnover pool:

$$f(x_2(t)) = \frac{1}{k}(1 - e^{-k\gamma x_2(t)})$$

- Linearize f

$$f(x_2(t)) = f(x_2^0) + f'(x_2^0)(x_2(t) - x_2^0)$$
$$= \frac{1}{k}(1 + exp(-k\gamma x_2^0)) + \gamma exp(-k\gamma x_2^0)(x_2(t) - x_2^0)$$

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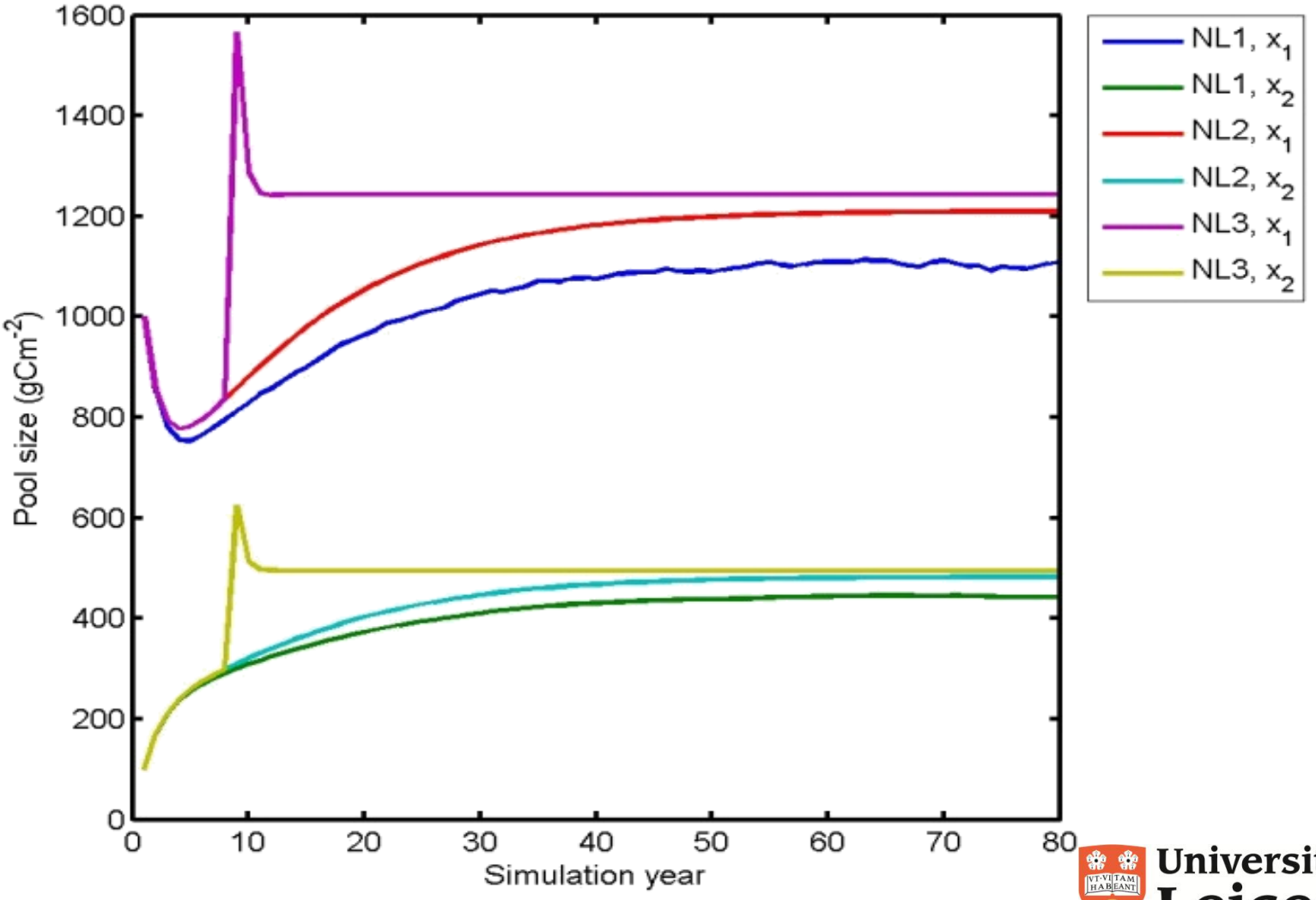
Results

	NL1	NL2	NL3
Short description	Numerical method, ode 45	Explicit forward iteration with A and g	8 years NL2, then 73 years of using $\Psi(T,0)$ and $\tilde{x}$ using eig(T) and inv(T), solving for qss
No years for qss estimate, yr_{qss}	5000	5000	80
C_{Tot} of qss (gCm^{-2})	48993.1	53149.7	54479.0
No of years until $\Delta x_T < 0.01$, $yr_{0.01}$	1020	999	11
C_{Tot} ($yr_{0.01}$) (gCm^{-2})	49489.6	53628.0	54608.1
dCTot/dt ($gCm^{-2}yr^{-1}$) in year $yr_{0.01}$	9.0	2.7	123.2
time for $yr_{0.01}$ yrs (s)	267.7	378.8	15.6

Results

- Fast pools x_1 & x_2



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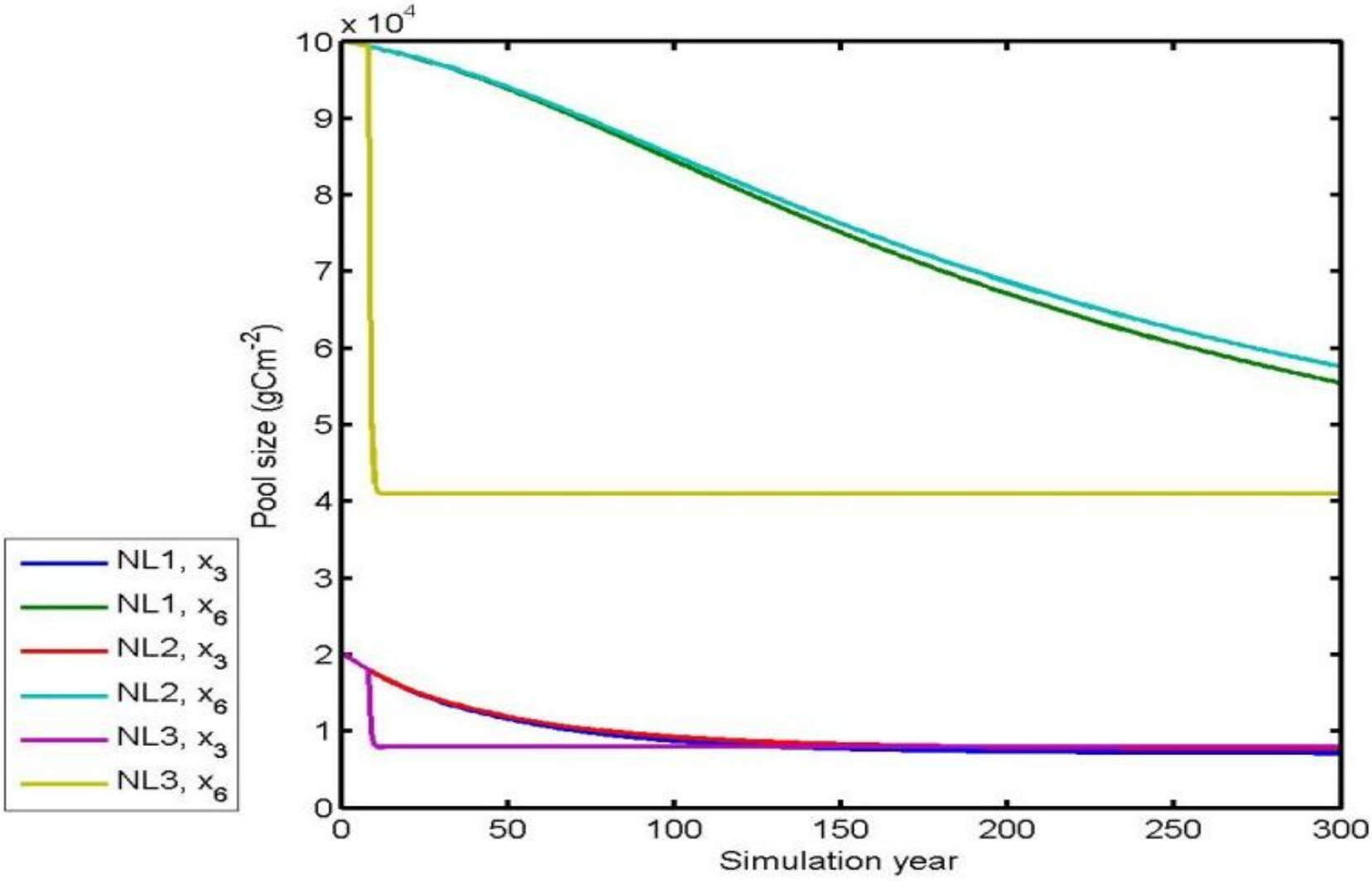
Results

- Slow pools x_3 & x_3



Results

- Different methods deliver different steady states
- New method needs for steady state of linearized system less than 5% of the time required by 1st order forward method
- Steady state difference between 1st order forward and new method is due to linearization
- Running 210 years 1st order forward method after steady state estimation from new method for steady state of 1st order forward method still needs less than 25% of the time required for 1st order forward method

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Summary

- Steady state of time varying linear system can be calculated directly from data of one simulation year
- Time savings are 99 – 99.99% for linear systems
- Virtually no additional resources (computational or manpower) required
- Can be applied to non-linear systems work through linearizations of the carbon system
- Difference between linearized and original system can be reduced by additional simulations
- Time savings of more than 75% possible
- Real time savings very large as A and g expensive

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Outlook

- For new method model carbon cycle system could be split into linear and nonlinear part
- New method facilitates exploration of climate carbon cycle feedbacks
- Allows improved data assimilation
- Currently being implemented in JULES
- Summited to GMDD very soon

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