Technical Advance: The Geometric-Series Solution (GSS) to spin up soil organic matter (SOM) pools

Wong, H.M., Hillier, J., Clark, D., Smith, J. and Smith, P.

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Introduction

- **Major application of SOM models:**
  - Initialize SOM pools using a set of driving data
  - Apply another set of driving data to study the impact on SOM pools

- **Pools of first-order based SOM models have equilibrium**
  - Difficult to interpret the model results when SOM pools are not in equilibrium
  - The most intrinsic way to solve the equilibrium problem is to reiterate SOM models for numerous times to get the equilibrium values
How to spin up SOM pools?

- Iterate the model
  - The computer is going to sweat!
  - Possible to use a longer time step (say, a year) to reduce the overhead, but it smooths the diurnal / seasonal fluctuation.

- Recent publications (Lardy et al., 2011 and Xia et al., 2012) suggested matrix methods to approximate the initialized SOM values.
  - The methods significantly reduce the number of iterations (still need some)
  - The methods are approximate methods, error analysis may be necessary
Here we present an analytical method – the GSS method – which solves the spin-up problem:

- The solution is exact, no need for error analysis
- Only one loop is required
- Only implement one equation

Major assumptions (limitations):

- Users spin up SOM pools by repeating a limited set of driving data (e.g. you repeat a 30-year long-term average driving data for 100 cycles to simulate a 3000-year model run)
- Modelled plant input and soil climate data do not change from one cycle to another cycle
- Inputs to each SOM pool can be derived from plant inputs analytically. No guarantee that it is applicable to all SOM models, but yes for the JULES-RothC / JULES-ECOSSE model.
- The SOM models are based on first-order-difference equations

\[
\frac{\delta \text{SOM}_p}{\delta t} = \left( \text{Pool\_input}_p + \text{SOM}_{(t-1)} \right) (1 - e^{-k_p t})
\]

\[k_p = c_p \times 1_t \times 2_t \ldots \times b_t\]
Appendix: ECOSSE C components

INPUTS (NPP, Land use Type)

RPM

DPM

CH

CO

BIO

HUM

Decomposing materials

CO₂

CH₄

DOC

Pool inputs

Decomposition

f (soil moisture, water table, hydraulic parameters)

f (soil parameters)

Temperature

Texture

pH

f (soil temperature)
Toolkits

Toolkit 1 – Sum of geometric series:

\[ \sum_{n=1}^{N-1} e^{nx} = S - e^0 = \frac{(1 - e^{nx})}{(1 - e^x)} - 1 \quad (3) \]

Toolkit 2 – \( e^{-k_a} \):

\[ e^{-k_a} = e^{-k_1} e^{-k_2} \ldots e^{-k_n} \quad (4) \]
Array of non-decomposed pool inputs during iteration

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<td>$e^{-k_{mn}}$</td>
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<td>$e^{-k_{mn}}$</td>
<td>$l_{mn}e^{-k_{mn}}$</td>
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Derivation of column sum I

\[ \text{colSums}(n) = l_{mn}(e^{-k_{mn}}) \]  
\[ + \ldots \]  
\[ + l_{2n}(e^{-k_{2n}})(e^{-k_{m1}} e^{-k_{m2}} \ldots e^{-k_{mn}}) \]  
\[ + l_{1n}(e^{-k_{1n}})(e^{-k_{21}} e^{-k_{22}} \ldots e^{-k_{2n}}) \ldots (e^{-k_{m1}} e^{-k_{m2}} \ldots e^{-k_{mn}}) \]

As plant inputs and soil climate data are the same across spin-up cycles, drop the first subscript:

\[ \text{colSums}(n) = l_{n}(e^{-k_{n}}) \]  
\[ + \ldots \]  
\[ + l_{n}(e^{-k_{n}})(e^{-k_{1}} e^{-k_{2}} \ldots e^{-k_{n}}) \]  
\[ + l_{n}(e^{-k_{n}})(e^{-k_{1}} e^{-k_{2}} \ldots e^{-k_{n}}) \ldots (e^{-k_{1}} e^{-k_{2}} \ldots e^{-k_{n}}) \]

Factor \( l_{n}(e^{-k_{n}}) \) out and use Toolkit 2:

\[ \text{colSums}(n) = l_{n}(e^{-k_{n}})(1 + e^{-k_{a}} + e^{-2k_{a}} + \cdots + e^{-mk_{a}}) \]  
\[ (7) \]
Derivation of column sum II

Use Toolkit 1:

\[ \text{colSums}(n) = I_n(e^{-kn})(1 - e^{-(m+1)ka})/(1 - e^{-ka}) - 1) \]  

(8)

Sum across columns:

\[ \text{SOM}_n = \sum_{n=1}^{n} \text{colSums}(n) \]  

(9)
Figure: Iterative runs of SOM pools and the results calculated by the GSS method.
Conclusion

- Based on mild assumptions, the GSS method was derived to initialize SOM pools.
- Application:
  - Spin up the JULES model and generate modelled data of plant inputs and soil climate.
  - Analytically find out the relationship between plant inputs and input of SOM pools.
  - Apply the formulae to calculate column sum of a timestep across spin-up cycles, then sum across the timesteps to get the SOM pool value.

\[
\text{colSums}(j) = l_j(e^{-k_j} e^{-k_{j+1}} \ldots e^{-k_n})(1 - e^{-(m+1)k_a})/(1 - e^{-k_a}) - 1
\]