Analytical determination of the carbon cycle equilibrium in coupled climate and carbon cycle models

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Outline

• Motivation
• Some maths background
• Application
• Results
• Summary
• Outlook
Motivation

• Carbon cycle models: predict future carbon (C) fluxes
• Simulation started with some initial state of the C pools (C in live biomass, litter, soil)
• Simulation results should reflect the response of the C system to the environmental variability during the simulation period
• Selection of initial state of C pools is crucial to avoid spurious C fluxes that result from adjustment of the system to “inappropriate” initial state
• Initial state often chosen as steady state of system (equilibrium)
Motivation

• **Equilibrium**

• **Difficult** to determine for a complex system

• Run model with the same boundary conditions and forcing for a long time and wait for convergence to steady state (100s to 1000s of years): **Expensive**

• Not a good initial value for simulations from now into the future, rather for, say 19th century: simulation into the future needs to capture current disequilibrium of the C system from the past 150-200 years

• Need another 150-200 years simulation years - expensive
Motivation

• Equilibrium
• Expensive
• Limits exploration of carbon cycle system
• Data assimilation (data assimilation of net C flux needs well established C state of the system)

• Direct determination from system dynamics?
• Needs to be computationally efficient
Maths background

• Carbon cycle systems are being represented by
  \[ \frac{dx}{dt}(t) = A(t, x) + g(t), x(t_0) = x_0, t \geq t_0 \]

• Frequently linear:
  \[ \frac{dx}{dt}(t) = A(t)x(t) + g(t), x(t_0) = x_0, t \geq t_0 \]

• \( x \) are the carbon pools, \( A \) environmental effects, \( g \) input, e.g. photosynthesis

• Soil respiration in JULES

\[
\begin{align*}
  d(DPM)/dt &= \alpha_{de} \Lambda_c - R_{DPM} \\
  d(RPM)/dt &= (1 - \alpha_{de}) \Lambda_c - R_{RPM} \\
  d(BIO)/dt &= 0.46 \cdot \beta R_s - R_{BIO} \\
  d(HUM)/dt &= 0.54 \cdot \beta R_s - R_{HUM}
\end{align*}
\]
Maths background

\[
\frac{dx}{dt}(t) = A(t)x(t) + g(t), x(t_0) = x_0, t \geq t_0
\]

- Solution:

\[
x(t) = \Psi(t, t_0)x_0 + \int_{t_0}^{t} \Psi(t, s) g(s) \, ds
\]

- \(\Psi(t, t_0)\) is the state transition matrix

- For constant A:

\[
\Psi(t, t_0) = \exp(A(t - t_0)) = e^{A(t-t_0)}
\]

- State transfer between time points

\[
\Psi(t_2, t_0) = \Psi(t_2, t_1)\Psi(t_1, t_0)
\]
Maths background

- Equilibrium:
- Steady state such that after one year the C pools are the same as at the beginning of the year:

\[ x_0 = x(t_0 + \omega) = \Psi(t_0 + \omega, t_0)x_0 + \int_{t_0}^{t_0+\omega} \Psi(t, s) g(s)ds \]

\( \omega \) period of one year

- So, for a steady state:

\[ x_0 = (I - \Psi(t_0 + \omega, t_0))^{-1} \int_{t_0}^{t_0+\omega} \Psi(t_0 + \omega, s) g(s)ds \]
Application

- Carbon cycle model is run time stepped with time step $\Delta t$
- $A$ and $g$ constant during time step: $A_i$ and $g_i$

\[ x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-s)} g(s) ds \]

- From one time step to the next:

\[ x_{i+1} = e^{A_i \Delta t} x_i + g_i \int_{t_i}^{t_{i+1}} e^{A_i(t-s)} ds \]

- Recursion
Application

• Resolve recursion:

\[ \Psi(t_0 + \omega, t_0) = \prod e^{A_i \Delta t} \]
\[ \int_{t_0}^{t_0 + \omega} \Psi(t_0 + \omega, s) g(s) \, ds = \sum g_i \int_{t_i}^{t_{i+1}} e^{A_i(t-s)} \, ds \]

• Computationally very efficient:
  
  – For n C pools add only one n x n matrix and one n vector to model code for each grid cell and PFT.
  
  – No need to output additional data or save any additional data in file for later calculations.
Application

Example

• Synthetic system

\[
A = \begin{pmatrix}
  a_{11} & 0 & 0 & 0 & 0 & 0 \\
  a_{21} & a_{22} & 0 & 0 & 0 & 0 \\
  a_{31} & 0 & a_{33} & 0 & 0 & 0 \\
  0 & a_{42} & 0 & a_{44} & 0 & 0 \\
  0 & 0 & a_{53} & 0 & a_{55} & 0 \\
  0 & 0 & 0 & a_{64} & a_{65} & a_{66}
\end{pmatrix}
\]

\[
a_{11} = \text{abs}(\max(10^{-5}, \sin(t \pi / nt) / nt))
\]
\[
a_{22} = 0.25 a_{11}; a_{21} = -0.5 a_{11}; a_{33} = 1/16 a_{11}; a_{31} = -0.5 a_{11};
\]

\[
g(t) = 5000 \times (\max(0, \sin(t \pi / nt) \times \sin(t \times 2 \pi / (24 \times 60) - \pi / 2)) / nt) \times 0 0 0 0 0 0^T
\]
## Results

<table>
<thead>
<tr>
<th>Case</th>
<th>Short description</th>
<th>Simulation years $\Delta x_T &lt; 0.01$</th>
<th>cputime in matlab (s)</th>
<th>$C_{Tot} \text{ von qss}$</th>
<th>$C_{Tot} \text{ when } \Delta x_T &lt; 0.01$</th>
</tr>
</thead>
<tbody>
<tr>
<td>I1</td>
<td>Numerical higher order solver, ode45</td>
<td>528</td>
<td>809</td>
<td>152883.70</td>
<td>151333.09</td>
</tr>
<tr>
<td>I2</td>
<td>Explicit forward iteration with $A$ and $g$</td>
<td>528</td>
<td>336</td>
<td>152814.02</td>
<td>151269.73</td>
</tr>
<tr>
<td>I3</td>
<td>$\Psi(T, 0)$ and $\tilde{x}$ determined using similarity transform, eig($T$) and inv($T$), iterated for all simulation years</td>
<td>528</td>
<td>2.4</td>
<td>152814.02</td>
<td>151269.72</td>
</tr>
<tr>
<td>I4</td>
<td>$\Psi(T, 0)$ and $\tilde{x}$ determined using similarity transform, eig($T$) and inv($T$), solved for qss</td>
<td>1</td>
<td>2.4</td>
<td>152814.02</td>
<td></td>
</tr>
</tbody>
</table>
Results

- "SOC" pool $x_6$
Results

• Different methods deliver the same steady state
• New approach needs less than 1% of the time of other methods for steady state
• With a more computationally complex A and g (e.g. JULES) absolute time savings would be very significant
Results

Non linear example

• Use linearization: Taylor-expansion

• Simple canopy integration:

\[ p_0 \ast (1-e^{-k\text{LAI}})/k \]

• \( p_0 \) top leaf photosynthesis

• Specific leaf area links to fast turnover pool:

\[ f(x_2(t)) = \frac{1}{k}(1 - e^{-k\gamma x_2(t)}) \]

• Linearize \( f \)

\[
\begin{align*}
  f(x_2(t)) &= f(x_2^0) + f'(x_2^0)(x_2(t) - x_2^0) \\
  &= \frac{1}{k}(1 + \exp(-k\gamma x_2^0)) + \gamma \exp(-k\gamma x_2^0)(x_2(t) - x_2^0)
\end{align*}
\]
## Results

<table>
<thead>
<tr>
<th>Short description</th>
<th>NL1</th>
<th>NL2</th>
<th>NL3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Numerical method, ode 45</td>
<td>Explicit forward iteration with A and g</td>
<td>8 years NL2, then 73 years of using $\Psi(T, 0)$ and $\tilde{x}$ using eig(T) and inv(T), solving for qss</td>
<td></td>
</tr>
<tr>
<td>No years for qss estimate, yr&lt;sub&gt;qss&lt;/yr&gt;</td>
<td>5000</td>
<td>5000</td>
<td>80</td>
</tr>
<tr>
<td>$C_{Tot}$ of qss (gCm$^{-2}$)</td>
<td>48993.1</td>
<td>53149.7</td>
<td>54479.0</td>
</tr>
<tr>
<td>No of years until $\Delta x_T &lt; 0.01$, yr&lt;sub&gt;0.01&lt;/yr&gt;</td>
<td>1020</td>
<td>999</td>
<td>11</td>
</tr>
<tr>
<td>$C_{Tot}$ (yr&lt;sub&gt;0.01&lt;/yr&gt;) (gCm$^{-2}$)</td>
<td>49489.6</td>
<td>53628.0</td>
<td>54608.1</td>
</tr>
<tr>
<td>$dC_{Tot}/dt$ (gCm$^{-2}$yr$^{-1}$) in year yr&lt;sub&gt;0.01&lt;/yr&gt;</td>
<td>9.0</td>
<td>2.7</td>
<td>123.2</td>
</tr>
<tr>
<td>Time for yr&lt;sub&gt;0.01&lt;/yr&gt; yrs (s)</td>
<td>267.7</td>
<td>378.8</td>
<td>15.6</td>
</tr>
</tbody>
</table>
Results

- Fast pools $x_1$ & $x_2$
Results

- Slow pools $x_3$ & $x_3$
Results

• Different methods deliver different steady states
• New method needs for steady state of linearized system less than 5% of the time required by 1\textsuperscript{st} order forward method
• Steady state difference between 1\textsuperscript{st} order forward and new method is due to linearization
• Running 210 years 1\textsuperscript{st} order forward method after steady state estimation from new method for steady state of 1\textsuperscript{st} order forward method still needs less than 25% of the time required for 1\textsuperscript{st} order forward method
Summary

• Steady state of time varying linear system can be calculated directly from data of one simulation year
• Time savings are 99 – 99.99% for linear systems
• Virtually no additional resources (computational or manpower) required
• Can be applied to non-linear systems work through linearizations of the carbon system
• Difference between linearized and original system can be reduced by additional simulations
• Time savings of more than 75% possible
• Real time savings very large as A and g expensive
Outlook

• For new method model carbon cycle system could be split into linear and nonlinear part
• New method facilitates exploration of climate carbon cycle feedbacks
• Allows improved data assimilation
• Currently being implemented in JULES
• Summitted to GMDD very soon

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